The World's Longest Proof of Tychonoff's Theorem

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ABSTRACT: This is a long but simple proof of Tychonoff's theorem. It is meant to mimic a Cantor diagonalization argument: Given an open cover $\mathcal{O}$ (of the product space $\mathbf{X}$) which has no finite subcover, the argument inductively picks a point in each coordinate space so that the resulting tuple is a point in the product space which is not, in fact, covered by $\mathcal{O}$ after all.

1: Example. *Sequential compactness is not preserved under arbitrary Cartesian product.* ♢

Proof. Let $\mathbb{S}$ be the set $\{-1,0,1\}^{\mathbb{N}}$. Set $Y := \{-1,0,1\}$ equipped with the discrete topology and let $X := Y^{\mathbb{S}}$ have product topology. Consider the sequence $(x_n)_{n=0}^{\infty} \subset \mathbf{X}$ defined by

$$x_n(s) \quad := \quad s(n)\,.$$

No subsequence of $(x_n)_1^{\infty}$ is convergent. For suppose $f\colon \mathbb{N} \to \mathbb{N}$ is a strictly increasing function and consider the subsequence $(x_{f(k)})_{k=0}^{\infty}$. Let s denote *the* element of $\mathbb{S}$ defined by

$$s(n) := \begin{cases} 0 & \text{if } n \notin \text{Range}(f); \\ [\text{-}1]^k & \text{if } n = f(k). \end{cases}$$

Now $(x_{f(k)}(s))_{k=0}^{\infty}$ is *not* a convergent sequence in Y, as

$$x_{f(k)}(s) \stackrel{\text{def}}{=} s(f(k)) \;=\; [\text{-}1]^k$$

shows with a vengeance. ♦

We establish some preliminary lemmas.

2: Lemma. *If X and Y are compact spaces then $X \times Y$ is compact.* ♢

Proof. Let $\mathcal{O}$ be an open cover of $X \times Y$. By replacing each $U \in \mathcal{O}$ by the collection of open rectangles $A{\times}B \subset U$ we may without loss of generality assume that every member of $\mathcal{O}$ is a rectangle.

Fix $x \in \mathbf{X}$. Since Y is compact, so is $\{x\}{\times}Y$. Thus there is a finite subcollection $\mathcal{F} \subset \mathcal{O}$ such that

$$\{x\} \times Y \;\subset\; \bigcup\{A \times B \mid A \times B \in \mathcal{F} \;\&\; A \ni x\}\,.$$

Define $I(x)$ to be the open set $\bigcap\{A \mid A \times B \in \mathcal{F}\}$. Evidently

$$\{x\} \times Y \;\subset\; I(x) \times Y \;\subset\; \bigcup\{A \times B \mid A \times B \in \mathcal{F}\}\,.$$

By compactness of X there exists a finite subset $\text{Fin} \subset \mathbf{X}$ such that $\bigcup_{x \in \text{Fin}} I(x)$ equals X. Writing the above $\mathcal{F}$ as $\mathcal{F}_x$ we have that

$$\{A \times B \mid x \in \text{Fin} \;\&\; A \times B \in \mathcal{F}_x\}$$

is a finite subcover of $\mathcal{O}$. ♦

Definition. Say that an open set $A \subset X \times Y$ is ***X-open*** if it is of the form $\mathbf{X}' \times Y$, where $\mathbf{X}'$ is an open subset of X. For $A \subset X \times Y$, let $\mathbb{P}A$ denote the unique maximum X-open subset of A, that is, the union of all X-open subsets of A. □

Henceforth letters α, β, λ and γ denote ordinals which are less than Λ, another ordinal.

3: Lemma. *Suppose Y is compact and Λ is some ordinal.*

- **i:** *If $\{x\}{\times}Y \subset U$, where U is an open subset of $X{\times}Y$, then $\{x\}{\times}Y \subset \mathbb{P}U$.*
- **ii:** *Suppose $\{U^\beta \mid \beta \in \Lambda\}$ is an increasing chain of open subsets of $X{\times}Y$. Then*

$$\bigcup_{\beta \in \Lambda} \mathbb{P}(U^\beta) \;=\; \mathbb{P}\Big(\bigcup_{\beta \in \Lambda} U^\beta\Big)\,. \qquad ♢$$

Remark. Above, we used the symbol "$\bigcup$" (increasing-union symbol) to indicated an *increasing* union, i.e, $\alpha < \beta$ implies $U^\alpha \subset U^\beta$.

Note that (3ii) need not hold if the U^β-sets fail to be open. For example, let Y be a singleton, X be the reals and $(q_n)_1^{\infty}$ be an enumeration of the rationals. Now

$$U^n := \mathbb{R} \smallsetminus \{q_n, q_{n+1}, q_{n+2} \ldots\}$$

as no interior, $\mathbb{P}(U^n) = \varnothing$. And $U^1 \subset U^2 \subset \ldots$. Yet $\mathbb{P}(\bigcup_n U^n) = \mathbb{P}(\mathbb{R}) = \mathbb{R}$. □

Proof of (i). For each $y \in Y$ there exists $A(y)$ an open subset of X, and $B(y)$ an open subset of Y, for which the ordered-pair

$$\langle x, y\rangle \;\in\; A(y){\times}B(y) \;\subset\; U\,.$$

By compactness of Y, there is a finite subset $F \subset Y$ such that $Y = \bigcup_{y\in F} B(y)$. Now $I := \bigcap_{y\in F} A(y)$ is an open set owning x. Therefore

$$I{\times}Y \;\subset\; \bigcup_{y\in F} A(y){\times}B(y) \;\subset\; U\,,$$

and so $\{x\}{\times}Y \subset I{\times}Y \subset \mathbb{P}U$. ♦

Proof of (ii). (In the sequel, the convention is that any unconstrained indexing ordinal, in this case "β", ranges over all $\beta \in \Lambda$.) Since $\mathbb{P}U^\beta \subset U^\beta$, the union $\bigcup_\beta \mathbb{P}U^\beta$ is an X-open subset of $\bigcup_\beta U^\beta$. Thus we have the "$\subset$" direction of (ii).

Conversely, suppose $x \in \mathbf{X}$ is such that $\{x\}{\times}Y \subset \mathbb{P}(\bigcup_\beta U^\beta)$. Then $\{x\}{\times}Y \subset \bigcup_\beta U^\beta$ and so $Y = \bigcup_\beta B^\beta$, where B^β denotes the subset of Y such that $\{x\}{\times}B^\beta = [\{x\}{\times}Y] \cap U^\beta$. This cross-section B^β is an open subset of Y by definition of the product topology. By compactness of Y, there exists a finite set $F \subset \Lambda$ such that $Y = \bigcup_{\beta\in F} B^\beta$. But $\{U^\beta \mid \beta < \Lambda\}$ is an increasing collection and therefore $\{B^\beta\}_\beta$ is a collection which increases. Hence

$$\bigcup_{\beta\in F} B^\beta \;=\; B^\mu\,, \qquad \text{where } \mu := \operatorname{Max}(F) \overset{\text{note}}{<} \Lambda\,.$$

Thus $\{x\}{\times}Y \subset U^\mu$ and so, (i) tells us, $\{x\}{\times}Y \subset \mathbb{P}(U^\mu)$. Consequently $\bigcup_\beta \mathbb{P}(U^\beta) \supset \{x\}{\times}Y$. This gives the "$\supset$" direction in (ii). ♦

The induction proof

For each ordinal α let Y^α be some compact space and let Λ be the minimal ordinal such that, for the sake of contradiction, the product space $\mathbf{X} := \bigotimes_{\alpha\in\Lambda} Y^\alpha$ fails to be compact. By lemma 2 we know that Λ must be a limit ordinal. For each ordinal $\lambda \le \gamma$ define the product space

$${}_\lambda\mathbf{X}_\gamma := \bigotimes \{Y^\alpha \mid \lambda \le \alpha < \gamma\}\,.$$

Let $\mathbf{X}_\alpha$ abbreviate ${}_0\mathbf{X}_\alpha$ and let ${}_\alpha\mathbf{X}$ stand for ${}_\alpha\mathbf{X}_\Lambda$. A set $A \subset \mathbf{X}$ is α-***open*** if it is of the form $B \times {}_\alpha\mathbf{X}$ for some open $B \subset \mathbf{X}_\alpha$. Let $\mathbb{P}^\alpha(A)$ denote the unique maximum α-open subset of A. If $\lambda < \gamma$ then any λ-open set is *a fortiori* γ-open.

4: Proposition. *For open sets $A, B \subset \mathbf{X}$ and ordinals α, λ, γ:*

a: $A \subset B \implies \mathbb{P}^\alpha A \subset \mathbb{P}^\alpha B$.

b: $\lambda \le \gamma \implies \mathbb{P}^\lambda(A) \subset \mathbb{P}^\gamma(A)$.

c: $\lambda \le \gamma \implies \mathbb{P}^\lambda \circ \mathbb{P}^\gamma = \mathbb{P}^\lambda$. ◊

Proof of (c). Applying $\mathbb{P}^\lambda$ to both sides of the inclusion $\mathbb{P}^\gamma(A) \subset A$ yields $\mathbb{P}^\lambda(\mathbb{P}^\gamma(A)) \subset \mathbb{P}^\lambda(A)$, by part (a). Conversely, applying $\mathbb{P}^\lambda$ to both sides of the conclusion of (b) yields

$$\mathbb{P}^\lambda(\mathbb{P}^\gamma(A)) \;\supset\; \mathbb{P}^\lambda(\mathbb{P}^\lambda A) \;=\; \mathbb{P}^\lambda(A)$$

by (a). Thus $\mathbb{P}^\lambda(\mathbb{P}^\gamma A) = \mathbb{P}^\lambda(A)$. ♦

Setting up a contradiction

We now proceed to contradict the assumption that Λ was the smallest ordinal such that $\mathbf{X}_\Lambda$ is non-compact. Henceforth let α, β, γ be ordinals ranging over all of $[0\,..\,\Lambda)$.

Simplifying the open cover

Presume that $\mathcal{O}$ is an open cover of $\mathbf{X}$. Without loss of generality we can replace each $U \in \mathcal{O}$ by all of the finite dimensional open boxes which are subsets of U. So now, for each set $E \in \mathcal{O}$, there exists $\alpha < \Lambda$ such that E is α-open. Thus $\bigcup_\alpha V^\alpha = \bigcup(\mathcal{O})$, where V^α is the α-open set

$$V^\alpha := \bigcup\{E \in \mathcal{O} \mid E \text{ is } \alpha\text{-open}\}.$$

It suffices to show, for some α, that $V^\alpha = \mathbf{X}$; for then $\{E \in \mathcal{O} \mid E \text{ is } \alpha\text{-open}\}$ is effectively an open cover of the compact space $\mathbf{X}_\alpha$. WLOG, then, $\mathcal{O} = \{V^\alpha \mid \alpha \in \Lambda\}$. The improvement is that, now, $\mathcal{O}$ is an increasing chain of sets.

A second simplification

Define $U^\alpha := \biguplus_{\beta\in\Lambda} \mathbb{P}^\alpha(V^\beta)$. By the foregoing proposition, $\{U^\alpha\}_{\alpha\in\Lambda}$ is an increasing chain. Since

$$U^\alpha \supset \mathbb{P}(V^\alpha) \supset V^\alpha,$$

we have that $\bigcup_\alpha U^\alpha = \bigcup(\mathcal{O})$. Now suppose we could exhibit an $\alpha < \Lambda$ such that $U^\alpha = \mathbf{X}$. Then the collection

$$\mathcal{C} := \{\mathbb{P}^\alpha(V^\beta) \mid \beta \in \Lambda\}$$

is an open cover of $\mathbf{X}$. But the "$\mathbf{X}_\alpha$ component" of the members of $\mathcal{C}$ form an open cover of the compact space $\mathbf{X}_\alpha$; thus it has finite subcover. The corresponding sets in $\mathcal{C}$ consequently form a finite cover of $\mathbf{X}$. Thus the corresponding finite collection of sets V^β is then a subcover of $\mathcal{O}$.

Since we need but show that some U^α equals $\mathbf{X}$, we can now WLOG assume that our cover is

5: $\mathcal{O} = \{U^\alpha \mid \alpha\in\Lambda\}$, *which is an increasing chain of α-open sets.*

The improvement of the $\{U^\alpha\}_\alpha$ over the $\{V^\alpha\}_\alpha$ is that

5b: $$\forall\lambda \le \gamma: \quad U^\lambda = \mathbb{P}^\lambda(U^\gamma).$$

This follows immediately from the computation

$$U^\lambda \stackrel{\text{def}}{=} \biguplus_\beta \mathbb{P}^\lambda(V^\beta) = \biguplus_\beta \mathbb{P}^\lambda(\mathbb{P}^\gamma(V^\beta)) \quad \text{by (4c)}$$
$$= \mathbb{P}^\lambda\Big(\biguplus_\beta \mathbb{P}^\gamma(V^\beta)\Big) \quad \text{from Lemma (3ii)}$$
$$\stackrel{\text{def}}{=} \mathbb{P}^\lambda(U^\gamma).$$

Also note that

5c: *For each limit ordinal α:* $$\bigcup_{\lambda\in\alpha} U^\lambda = U^\alpha.$$

For let U represent U^α, but viewed as an open subset of $\mathbf{X}_\alpha$. So in light of (5b) we need but show that $\bigcup_{\lambda\in\alpha}\mathbb{P}^\lambda U \supset U$; the "$\subset$" direction being trivial. Fix an $x \in U$. By definition of the product topology, $x \in \bigotimes_{\lambda\in\alpha} \hat{Y}^\lambda \subset U$ where $\hat{Y}^\lambda$ is an open subset of Y^λ with $\hat{Y}^\lambda = Y^\lambda$ for all λ outside of some finite index set $F \subset \alpha$. Thus $\mu := \operatorname{Max}(F)$ is less than α. Hence $x \in \mathbb{P}^\mu(U)$, completing the argument.

Obtaining a contradiction

Regard the symbol x_α, below, as a point in $\mathbf{X}_\alpha$. We shall inductively construct such points, by extension, via the following.

$M(\alpha)$: *There is an ordered set $x_\alpha = \langle y^\lambda \mid \lambda \in \alpha\rangle$, with each y^λ a point in the compact space Y^λ, such that*
$\{x_\alpha\} \times_\alpha \mathbf{X} \subset \mathbf{X} \setminus U^\alpha$
and satisfying the consistency condition:
$\lambda < \alpha \implies x_\lambda \subset x_\alpha$.

Establishing this will complete the proof of Tychonoff's theorem by showing that $\mathcal{O}$ did not cover $\mathbf{X}$ after all: For define a point $x := \bigcup_{\alpha\in\Lambda} x_\alpha$; by the consistency condition this is a point in $\mathbf{X}$. Thus

$$x \in \{x_\alpha\} \times_\alpha \mathbf{X} \subset \mathbf{X} \setminus U^\alpha,$$

for each $\alpha \in \Lambda$. Hence $x \in \mathbf{X} \setminus \bigcup_\alpha U^\alpha = \mathbf{X} \setminus \bigcup(\mathcal{O})$.

Proof of $M(\alpha)$. Argue by induction on α. If α is a limit ordinal then define x_α to be $\bigcup_{\lambda\in\alpha} x_\lambda$. Then, by $M(\alpha)$, the point $x_\alpha \in \{x_\lambda\} \times_\lambda \mathbf{X}$ and so

$$\{x_\alpha\} \times_\alpha \mathbf{X} \subset \{x_\lambda\} \times_\lambda \mathbf{X} \subset \mathbf{X} \setminus U^\lambda.$$

Hence $\{x_\alpha\} \times_\alpha \mathbf{X} \subset \mathbf{X} \setminus \bigcup_{\lambda\in\alpha} U^\lambda$, which equals $\mathbf{X} \setminus U^\alpha$ by (5c).

Conversely, suppose α is a successor ordinal $\alpha = \beta+1$. Were $\{x_\beta\} \times_\beta \mathbf{X}$ a subset of $U^{\beta+1}$ then, by (3i),

$$\{x_\beta\} \times_\beta \mathbf{X} \subset \mathbb{P}^\beta(U^{\beta+1}) = U^\beta.$$

This contradicts $M(\beta)$. Consequently, there exists a point $y \in Y^\beta$ such that $\langle x_\beta, y\rangle \times_{\beta+1} \mathbf{X}$ is not a subset of $U^{\beta+1}$; hence (since $U^{\beta+1}$ is $[\beta+1]$-open) it is disjoint from $U^{\beta+1}$. So defining $x_{\beta+1}$ to be $\langle x_\beta, y\rangle$ establishes $M(\beta+1)$. Whew! ♦

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